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Semi-Combinants as Concomitants of Affiliants.

By Henry S. White.

Introduction.

The theory of invariants received its natural extension and completion in that of seminvariants. So from the study of combinants it would be expected that Sylvester should have advanced to the investigation of semi-combinants. Since, however, a combinant differs from other invariants only in being invariant with respect to two or more independent systems of linear transformations, such an extension promised little novelty of method. I allude to this possibility only in order to propose a larger use for the term *semi-combinant.* In Sylvester's original formulation (1853) combinants are defined as "concomitants to systems of functions remaining invariable, not only when combinations of the variables are substituted for the variables, but also when combinations of the functions are substituted for the functions."* These have their place in the discussion of a system of forms or quantics, all of which have the same order in the variables. Where, however, a system of forms of unequal orders is to be discussed, there too arise concomitants which belong to the "system in its corporate capacity." But linear combination of the forms with constant multipliers is not possible, since homogeneity in the variables must be preserved. Accordingly the "system in its corporate capacity" will comprise compound forms, each form being modified by the addition of each form of lower order than itself, multiplied by an arbitrary form of suitable order. The coefficients of each such arbitrary form constitute the constants of an independent transformation. An invariant of one such transformation I propose to call a *semi-combinant* of the two forms concerned; an invariant of all such transformations, a *semi-combinant of the entire system.*

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* Cambridge and Dublin Math. Jour., Vol. VIII, p. 62.

For example, let the groundforms be two binary forms of orders m and n respectively, f_m and ϕ_n, where $m > n$. Borrowing Sylvester's terminology, let us call the compound form

$$F_m = f_m + R_{m-n} \cdot \phi_n$$

the conjunctive* of the system. The arbitrary constants are then the $m - n + 1$ coefficients of the arbitrary form R_{m-n}. Then every concomitant of the conjunctive F_m and the lower form ϕ_n will be a semi-combinant of f_m and ϕ_n, if it is independent of the arbitrary constants in R_{m-n}.

It is possible that the arbitrary form R_{m-n} can be equated to such a covariant as to make conjunctive F itself a semi-combinant. If this be done I call the particular value of the conjunctive a *semi-combinant groundform*. Having defined the term, we must inquire whether semi-combinant groundforms exist, and if so, how many there are that are linearly independent. I shall prove that if such a groundform exists it must satisfy identically a definite equation of condition found by elimination from a set of linear equations, and invariant in structure; an invariant equation, of which the well-known condition that a curve shall be apolar to a conic is a special example. Reversing now the order of things, I consider all groundforms that are included in the conjunctive of the system, and those of them that satisfy invariant equations of suitable order, linear in their coefficients, I designate as *affiliant* groundforms. I show that not only is every semi-combinant groundform an affiliant, but also every affiliant groundform is a semi-combinant. A given characteristic equation resolves into linear equations which determine the corresponding affiliant groundform, thus establishing directly the existence of semi-combinant groundforms. That their number is limited I show from the nature of the characteristic equations on the one hand; on the other hand, from the necessary structure of those groundforms as covariants. Both methods give the same upper limit, probably much too high, to their number. So much constitutes the main purpose of this paper.

Incidentally it will be noticed that apolarity is, in the binary domain and for quadric forms in any number of variables, a special case of the affiliant relation. Indeed the well-known connection between the theory of apolarity and the theory of combinants† finds its analogue in the relation of affiliants to

* Cambridge and Dublin Math. Jour., Vol. VIII, pp. 258, 259.

† See for example B. Igel: Ueber einige Anwendungen des Principes der Apolarität, Wiener Berichte XCII, pp. 1153–1194. Full references are given by F. Meyer in Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. I, p. 254 seq.

semi-combinants. The range of possible applications of this general theory is as wide as could be wished, since the simultaneous system may include any number whatever of groundforms in any number of variables, of any orders, alike or different. That nearest at hand is probably the application to curves of double curvature. Toward this I offer elementary suggestions; and similarly with regard to the fundamental question concerning a reduced form-system of semi-combinants. Finally, I point out in what way the introduction of semi-combinants may be useful in the discovery of normal forms, in special cases, for Abelian integrals, and suggest the extension of the theory to connexes and other mixed forms.

§1. *Covariant Curve apolar to a Conic determined as a Semi-Combinant.*

To Rosanes and Reye is due principally the theory of mutually apolar curves or surfaces. Rosanes called such curves *conjugirt* in relation to each other. Lindemann later used the still less distinctive appellation *vereinigt liegend;* but the term *apolar*, proposed by Reye, seems to be in use at present to the exclusion of both the others. Its meaning may be defined briefly in algebraic language as follows. If a locus of order n and a locus of class k be given by the equations

$$a_x^n = 0 \text{ and } u_\alpha^k = 0,$$

these loci are mutually apolar when the covariant of order $n - k$ (or contravariant of class $k - n$) linear in the coefficients of both equations is identically zero; i. e. when

$$a_\alpha^k a_x^{n-k} \equiv 0 \text{ if } n \geqq k,$$

or

$$a_\alpha^n u_\alpha^{k-n} \equiv 0 \text{ if } k \geqq n,$$

for all values of the variables (x) or (u). Inasmuch as each locus has both order and class, Reye disclosed an important theorem when he proved that the apolar relation is reciprocable; or, in other words, that if two loci are apolar, their polar reciprocals also are apolar.*

The study of apolarity received fresh impetus when Lindemann† found it available in the discussion of binary quantics by the aid of rational curves. A

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* Th. Ruge: Ueber algebraische Flächen, die zu einander apolar sind, Journal für r. u. a. Mathematik, Bd. 79, pp. 159-175.

† F. Lindemann: Sur une représentation géométrique des covariants des formes binaires, Bull. S. M. F., t. V, pp. 113-126.

binary form of even order $2n$ is transformed readily into a ternary quantic of order n, and thus its zeros (*roots* of the corresponding equation), which are ordinarily represented by $2n$ points on a line, come to be represented by the $2n$ points where a conic is met by a curve of order n. The equations of transformation determine the equation of the conic, the transformed quantic equated to zero determines the intersecting n^{ic}. *Lindemann showed that this n^{ic} is always apolar to the conic.* Now if the conic were given arbitrarily, and the $2n$ points upon it, the problem of finding the corresponding binary $2n^{ic}$ would require the discovery of a curve of n^{th} order intersecting the conic in the given points, and apolar to the conic. While through the $2n$ points there pass not only one but an infinity of curves of order n, only one of them is, in general, apolar to the conic. Lindemann's theorem may be stated thus:

Every complete intersection-system of points on a conic determines one, and only one, curve apolar to the conic, meeting the conic in those points and in no others.

Representing the conic and any curve which meets it in the given complete intersection-system by the equations

$$a_x^2 = 0 \text{ and } \alpha_x^n = 0,$$

Lindemann determined the apolar n^{ic} through the intersection-system as a rational covariant of these two curves.* Its equation is necessarily of the form

$$A_x^n = \pi_0 . \alpha_x^n + \pi_x^{n-2} a_x^2 = 0, \tag{1}$$

where π_x^{n-2} is some quantic of order $n-2$, π_0 a constant. That A_x^n is a covariant is obvious, assuming that it is uniquely determinate, from the invariant character of its equation of condition, viz.

$$(abA)^2 A_x^{n-2} \equiv 0. \tag{2}$$

Instead of reproducing Lindemann's derivation of this covariant A_x^n, I will give a more rapid derivation by the aid of its semi-combinant property.

There is but one quantic A_x^n satisfying the equation of apolarity; for, substituting the form (1) in equation (2) we have

$$\frac{n(n-1)}{2}\pi_0 (ab\alpha)^2 \alpha_x^{n-2} + \frac{(n-2)(n-3)}{2}(ab\pi)^2 \pi_x^{n-4} c_x^2$$
$$+ 2(n-2)(ab\pi)(abc)\pi_x^{n-1} c_x + (abc)^2 \pi_x^{n-2} \equiv 0,$$

* F. Lindemann: Sur une représentation géométrique des covariants des formes binaires, 2[me] note, Bull. S. M. F., t. VI, p. 195-208.

or after reduction of the third term,

$$\frac{n(n-1)}{2}\pi_0(ab\alpha)^2\alpha_x^{n-2}+\frac{(n-2)(n-3)}{2}(ab\pi)^2\pi_x^{n-2}c_x^2 + \frac{2n-1}{3}(abc)^2\pi_x^{n-2}\equiv 0. \qquad (3)$$

On separating this identical equation into its constituents, the latter would be linear in π_0 and the coefficients of π_x^{n-2}. Hence they are solved by a single system of values, or else by an indefinite number. Obviously equation (3) gives a recurrent method for determining the quantic π_x^{n-2} with its first term $\frac{3n(n-1)}{2(2n-1)}\cdot\pi_0\cdot\frac{(ab\alpha)^2\alpha_x^{n-2}}{(abc)^2}$ perfectly determinate. Accordingly, there is one, and only one, apolar quantic A_x^n.

This covariant A_x^n must be a semi-combinant of α_x^n and a_x^2. As it is unique, and its determining equation (2) is independent of the quantic α_x^n, it can depend only upon the given intersection-system, and that remains unchanged when the curve $\alpha_x^n=0$ is replaced by any curve of the system

$$\alpha_x^n+u_x^{n-2}.a_x^2=0,$$

where u_x^{n-2} denotes a quantic of order $n-2$ with independently variable coefficients. The covariant A_x^n remains unaltered therefore by the substitution of $\alpha_x^n+u_x^{n-2}a_x^2$ for α_x^n; that is, the covariant A_x^n is a semi-combinant.

As this semi-combinant is linear in the coefficients of α_x^n, its terms can contain symbolic factors of only the following types:

$$a_x^2,\ (abc)^2,\ (ab\alpha)^2,\ \alpha_x^{n-2i} \qquad (i=0,1,2,\ldots).$$

This is readily seen by the aid of ordinary theorems on ternary quadrics. I may therefore write the covariant as follows, using undetermined coefficients λ_0, λ_1, λ_2, etc., and denoting by Δ the discriminant $(abc)^2$, by α_Δ^2 the factor $(\alpha ab)^2$:

$$A_x^n=\lambda_0\Delta^r.\alpha_x^n+\lambda_1.\Delta^{r-1}.\alpha_\Delta^2\alpha_x^{n-2}.a_x^2+\lambda_2\Delta^{r-2}.\alpha_\Delta^2\alpha_{\Delta'}^2\alpha_x^{n-4}.a_x^2b_x^2+\text{etc.} \qquad (4)$$

The number of constants λ_k is $\frac{n+1}{2}$ or $\frac{n+2}{2}$, as n is odd or even. To determine their values, Lindemann applied the equation of apolarity, (2). I will employ the condition that renders A_x^n a semi-combinant. Substituting in (4) for

α_x^n the conjunctive $a_x^n + u_x^{n-2}a_x^2$, the increment of A_x^n must vanish identically. From the first term of (4) comes the increment

$$\lambda_0 \Delta^r u_x^{n-2} a_x^2 .$$

From the second,

$$\frac{2}{n(n-1)} \lambda_1 \Delta^{r-1} \left\{ a_\Delta^2 u_x^{n-2} + 2(n-2) a_\Delta u_\Delta a_x u_x^{n-3} + \frac{(n-2)(n-3)}{1.2} . u_\Delta^2 u_x^{n-4} a_x^2 \right\} b_x^2$$

which reduces to

$$\frac{2}{n(n-1)} \lambda_1 \left\{ \frac{2n-1}{3} . \Delta^r u_x^{n-2} a_x^2 + \frac{(n-2)(n-3)}{1.2} . \Delta^{r-1} . u_\Delta^2 u_x^{n-4} . a_x^2 b_x^2 \right\} .$$

Similarly the third term gives the reduced increment,

$$\frac{2}{n(n-1)} \lambda_2 . \left\{ \begin{array}{l} \left(1 + 1 + \frac{2.2}{3} + \frac{4(n-4)}{3}\right) \Delta^{r-1} u_\Delta^2 u_x^{n-4} . a_x^2 . b_x^2 \\ \qquad + \frac{(n-4)(n-5)}{1.2} \Delta^{r-2} u_\Delta^2 u_{\Delta'}^2 u_x^{n-6} a_x^2 b_x^2 c_x^2 \end{array} \right\} .$$

These suffice to show the law of the coefficients. We find on collecting terms and equating to zero,

$$\lambda_0 + \frac{2(2n-1)}{3n(n-1)} \lambda_1 = 0,$$

$$\lambda_1 + \frac{2.2.(2n-3)}{3(n-2)(n-3)} \lambda_2 = 0, \text{ etc.}$$

Hence

$$\lambda_1 = -\frac{3n(n-1)}{2(2n-1)} \lambda_0,$$

$$\lambda_2 = +\frac{3^2 . n(n-1)(n-2)(n-3)}{2.2^2(2n-1)(2n-3)} \lambda_0 .$$

Introducing additional factors in both terms of these fractions, we obtain them in the form

$$\lambda_1 = -\frac{3n . n(n-1)}{2n.(2n-1)} \lambda_0 = -\frac{3\, {}_nP_1 . {}_nP_2}{1 . {}_{2n}P_2} \lambda_0,$$

$$\lambda_2 = +\frac{3^2 . {}_nP_2 . {}_nP_4}{2!\, {}_{2n}P_4} \lambda_0 .$$

From the general equation of condition we obtain thus the value of λ_k in terms of λ_0:

$$\lambda_k = (-1)^k \frac{3^k\, {}_nP_k\, {}_nP_{2k}}{k!\, {}_{2n}P_k} \lambda_0 . \tag{5}$$

Here for abbreviation ${}_nP_k$ denotes the product

$$ {}_nP_k = n(n-1)(n-2)\ldots\ldots(n-k+1). $$

The value of λ_0 is arbitrary; we may assume $\lambda_0 = 1$. Omitting needless factors, the value of r may be taken less by unity than the number of possible terms

$$ r = \frac{n-1}{2} \text{ or } \frac{n}{2}. $$

Inserting these values of $\lambda_0 \ldots\ldots \lambda_k \ldots\ldots$, and r, in formula (4), we have the covariant and semi-combinant A_x^n expressed in terms of fundamental covariants of α_x^n and a_x^2.

The expression for A_x^n, obtained by its semi-combinant property, agrees with that derived by Lindemann from its apolarity to a_x^2. The foregoing discussion is a proof of Lindemann's theorem, since the semi-combinant property is in this case involved in the property of apolarity.

A precisely similar determination of a semi-combinant n^{ic} can be found when we interpret the forms α_x^n and a_x^2 as ternary, or quaternary, or m-ary. In each case the form obtained is identical with that given by the conditions of apolarity. The reason for this will appear in §§7 and 8 below.

§2. *The Differential Equations satisfied by Semi-Combinants.*

The name semi-combinant has been proposed above for certain invariants of a simultaneous system. Invariants (including, of course, covariants, etc.) are defined readily by differential equations which form a "complete system," and ordinary invariants thus come to be treated by Lie's method as invariants of a "group" of infinitesimal transformations. The same can be shown to be true of semi-combinants. In verification of this statement I need consider only two binary quantics as an example; it will appear that the discussion could be extended without difficulty to any number of quantics in any number of variables.

Denote by f_m and ϕ_n binary quantics of orders m and n respectively, and let $m > n$. To indicate with precision the several coefficients, let the terms of the two quantics be written without numerical factors,

$$ f_m = \sum_{i+k=m} (a_{i,\,k} x_1^i x_2^k), $$

$$ \phi_n = \sum_{i+k=n} (\alpha_{i,\,k} x_1^i x_2^k). $$

Then the four operators discussed in the theory of linear transformations are these:

$$\left.\begin{aligned}
W_1 &= \sum_{i+k=m}\left(i\,.\,a_{ik}\frac{\partial}{\partial a_{ik}}\right)+\sum_{i+k=n}\left(i\alpha_{ik}\frac{\partial}{\partial \alpha_{ik}}\right),\\
W_2 &= \sum_{i+k=m}\left(ka_{ik}\frac{\partial}{\partial a_{ik}}\right)+\sum_{i+k=n}\left(k\alpha_{ik}\frac{\partial}{\partial \alpha_{ik}}\right),\\
O_1 &= \sum_{i+k=m}\left((i+1)\,a_{i+1,\,k-1}\frac{\partial}{\partial a_{ik}}\right)+\sum_{i+k=n}\left((i+1)\,\alpha_{i+1,\,k-1}\frac{\partial}{\partial \alpha_{ik}}\right),\\
O_2 &= \sum_{i+k=m}\left((k+1)\,a_{i-1,\,k+1}\frac{\partial}{\partial a_{ik}}\right)+\sum_{i+k=n}\left((k+1)\,\alpha_{i-1,\,k+1}\frac{\partial}{\partial \alpha_{ik}}\right).
\end{aligned}\right\} \qquad (6)$$

These are here written as applied to invariants only; the variables neglected may be included, of course, by adjoining a linear quantic to the two under consideration. Of these four operators, the first two acting on an invariant reproduce it with a numerical factor, the third and fourth yield identically zero.

The additional operations entering into the theory of semi-combinants are educed from the definition. A semi-combinant of f_m and ϕ_n is an invariant which is unchanged when the coefficients of f_m are increased by the corresponding coefficients of $u_{m-n}\phi_n$; u_{m-n} denoting an arbitrary quantic of order $(m-n)$. Giving to u_{m-n} successively the $(m-n+1)$ linearly independent values λx_1^{m-n}, $\lambda x_1^{m-n-1}x_2, \ldots . \lambda x_2^{m-n}$, we shall obtain $(m-n+1)$ independent sets of increments to be applied to the coefficients of f_m, and shall find as many equations of condition for a semi-combinant.

To the coefficients

$$a_{m,\,0},\quad a_{m-1,\,1}, \ldots .\, a_{1,\,m-1}\ ,\ a_{0,\,m}\ ,$$

accrue the increments

$$\begin{array}{rllll}
(1) & \lambda\alpha_{n,\,0}, & \lambda\alpha_{n-1,\,1}, & \ldots .\ 0 & ,\ 0\ ,\\
(2) & 0 & ,\ \lambda\alpha_{n-0} & ,\ \ldots . & \\
 & \vdots & & & \\
(m-n) & 0 & ,\ 0 & ,\ \ldots .\ \lambda\alpha_{0,\,n} & ,\ 0\ ,\\
(m-n+1) & 0 & ,\ 0 & ,\ \ldots .\ \lambda\alpha_{1,\,n-1}, & \lambda\alpha_{0,\,n}.
\end{array}$$

Corresponding to these elementary sets of increments, construct the $(m-n+1)$ elementary differential operators:

$$\left.\begin{aligned}
S_{m-n,\,0} &= \alpha_{n,\,0}\frac{\partial}{\partial a_{m,\,0}} + \alpha_{n-1,\,1}\frac{\partial}{\partial a_{m-1,\,1}} + \dots + \alpha_{0,\,n}\frac{\partial}{\partial a_{m-n,\,n}} \\
&= \sum_{\left(\substack{i+k=n\\ l+k=m}\right)}\left(\alpha_{ik}\frac{\partial}{\partial a_{lk}}\right) \\
S_{m-n-1,\,1} &= \sum\left(\alpha_{i+1,\,k-1}\frac{\partial}{\partial a_{l,\,k}}\right), \\
S_{m-n-2,\,2} &= \sum\left(\alpha_{i+2,\,k-2}\frac{\partial}{\partial a_{l,\,k}}\right), \\
&\;\;\vdots \\
S_{0,\,m-n} &= \sum\left(\alpha_{i,\,k}\frac{\partial}{\partial a_{i,\,m-i}}\right).
\end{aligned}\right\} \tag{7}$$

The sets of increments tabulated above define each a substitution of functions of an arbitrary λ for the coefficients a_{ik}, and these substitutions must not alter the value of a semi-combinant F. Equating to zero the corresponding increments of F, there result the equations of condition:

$$\left.\begin{array}{c} S_{m-n,\,0}(F)=0,\;\; S_{m-n-1,\,1}(F)=0,\;\; S_{m-n-2,\,2}(F)=0, \\ S_{0,\,m-n}(F) \;\;=0. \end{array}\right\} \tag{8}$$

Referring to the definitions (7), it is apparent upon inspection that any two operators S are commutative, for the action of either upon the other is nil. Since then for all suffices i, k,

$$S_{m-n-i,\,i}\,S_{m-n-k,\,k} - S_{m-n-k,\,k}\,S_{m-n-i,\,i} \equiv 0,$$

the $(m-n+1)$ *differential equations* (8) *form a complete system, and the corresponding operations* S *define a group.** The four operations W_1, W_2, O_1, O_2 define a second group, as is well known. There remains to be examined then only the effect of permuting an operator of the first group and one of the second; finding this always expressible as a single operator of the first group, we shall conclude that all these operators together define a group comprehending the others as sub-groups. We find by easy reckoning the following facts:

$$\left\{\begin{aligned}
&S_{m-n,\,0}W_1 - W_1 S_{m-n,\,0} \quad\; = (m-n)\,S_{m-n,\,0}, \\
&S_{m-n-1,\,1}W_1 - W_1 S_{m-n-1,\,1} = (m-n-1)\,S_{m-n-1,\,1}, \\
&\;\;\vdots \\
&S_{m-n-r,\,r}W_1 - W_1 S_{m-n-r,\,r} = (m-n-r)\,S_{m-n-r,\,r};
\end{aligned}\right.$$

* S. Lie: Theorie der Transformationsgruppen, Erster Abschn., p. 107, Satz 4.

and similarly for W_2,

$$S_{m-n-r,\,r}\,W_2 - W_2\,S_{m-n-r,\,r} = r\,.\,S_{m-n-r,\,r}.$$

The results of permuting operators O with operators S are as follows:

$$\begin{cases} S_{m-n,\,0}\,O_1 - O_1\,S_{m-n,\,0} &= (m-n)\,S_{m-n-1,\,1}, \\ S_{m-n-1,\,1}\,O_1 - O_1\,S_{m-n-1,\,1} &= (m-n-1)\,S_{m-n-2,\,2}, \\ \vdots & \\ S_{1,\,m-n-1}\,O_1 - O_1\,S_{1,\,m-n-1} &= 1\,.\,S_{0,\,m-n}, \\ S_{0,\,m-n}\,O_1 - O_1\,S_{0,\,m-n} &= 0\,. \end{cases}$$

$$\begin{cases} S_{m-n,\,0}\,O_2 - O_2\,S_{m-n,\,0} &= 0\,, \\ S_{m-n-1,\,1}\,O_2 - O_2\,S_{m-n-1,\,1} &= 1\,.\,S_{m-n,\,0}, \\ S\,\vdots & \\ S_{1,\,m-n-1}\,O_2 - O_2\,S_{1,\,m-n-1} &= (m-n-1)\,S_{2,\,m-n-2}, \\ S_{0,\,m-n}\,O_2 - O_2\,S_{0,\,m-n} &= (m-n)\,S_{1,\,m-n-1}. \end{cases}$$

The completeness of the system of equations (6) and (8), thus verified, might have been inferred from the observation that all the operators involved are linear, of the general type known as polar operators, and comprised in an aggregate given by integral indices with definite limits. The above exact expression of the laws of the system shows, however, that as generators of the group, after the operations (6), any single one of the S_{ik} might have been selected. In other words,

The manifoldness of semi-combinants of two binary quantics is but one unit less than that of ordinary invariants.

Beside the extension, now evident, of this theorem to two quantics in any number of variables, there is another not remote. We may consider simultaneous invariants of more than two quantics; e. g. of three binary quantics

$$f_m,\ \phi_n,\ \psi_p \qquad (m \geqq n \geqq p),$$

and define as semi-combinants of the system (in that order) such invariants as are unaltered when these quantics are replaced by the three following:

$$f_m + U_{m-n}\phi_n + V_{m-p}\psi_p, \quad \phi_n + W_{n-p}\psi_p, \quad \psi_p,$$

in which U, V, W denote quantics of the proper orders with arbitrary coefficients. Here we can state the theorem:

The manifoldness of semi-combinants of three binary quantics—whose sequence

must be stated whenever two have the same order in the variables—is less by three units than that of their simultaneous invariants. Only three independent conditions beside those of ordinary invariants are needed to define them completely.

§3. *Semi-Combinant Groundforms defined, with Examples.*

The ternary quantic of order n apolar to a quadric, which is discussed in §1, is a semi-combinant of a given n^{ic} and quadric. It is a linear combination of covariant multiples of the n^{ic} and the quadric. The covariants are all linear in the coefficients of the n^{ic}. We shall characterize this apolar n^{ic} as a *semi-combinant groundform*, and make the following definition:

If two quantics have orders m and n respectively $(m > n)$, then any semi-combinant aggregate of covariants of order m containing severally one or the other quantic as factors, and linear in the coefficients of the m^{ic}, shall be called a semi-combinant groundform of the two quantics.

Denote two quantics by α_x^m and a_x^n, $(m > n)$. The only covariant linear in the coefficients of α_x^m, of order m, containing α_x^m as a factor is α_x^m multiplied by some invariant of a_x^n. Since other terms of the aggregate are to contain a factor a_x^n, the other factor of each must be a covariant of order $(m-n)$, linear in the coefficients of α_x^m. Since such covariants are to be mentioned often, I denote them uniformly by the initial letter of "linear"—namely, by $L_{m-n}^{(i)}$ or simply $L^{(i)}$. *The general type of a semi-combinant groundform is then*

$$A_x^m = I.\alpha_x^m + (\lambda_1 L' + \lambda_2 L'' + \dots + \lambda_k L^{(k)})\, a_x^n, \tag{9}$$

where k denotes the number of linearly independent covariants L, I some invariant of the quantic a_x^n. That such semi-combinant groundforms exist I will show by a few examples.

For a ternary m^{ic} and 2^{ic} the apolar m^{ic} of Lindemann is, as has been shown (§1), the only semi-combinant groundform. For it was found to be completely determined by the conditions employed, barring a factor which was an arbitrary power of the discriminant of the quadric.

For a binary m^{ic} and 2^{ic} the only semi-combinant groundform is that (conjugate) apolar to the quadric, as I have shown elsewhere.* The same considerations are valid for any number of variables if one of the forms is a quadric.

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*Reduction of the resultant of a binary quadric and n^{ic} by virtue of its semi-combinant property. Bulletin of the American Mathematical Society, Oct. 1894, pp. 11–15.

When a "reduced system" of invariants of two quantics is known, the determination of a semi-combinant groundform is usually easy, since only covariants $L^{(i)}_{m-n}$ enter into the problem, and the condition for a semi-combinant is applied by a single substitution. As first example, consider the binary quartic α_x^4 and cubic $a_x^3 = b_x^3 =$ etc. Following Gordan's method we find four linearly independent covariants of order 1, linear in the coefficients of α_x^4; four covariants L_1:

$$\begin{aligned} L^{\mathrm{I}} &= (a\alpha)^3\,\alpha_x, \\ L^{\mathrm{II}} &= (ab)^2(ac)(b\alpha)(c\alpha)^2\,\alpha_x, \\ L^{\mathrm{III}} &= (a\alpha)^3(bc)^2(b\alpha)\,c_x, \\ L^{\mathrm{IV}} &= (ab)^2(cd)^2(ae)(b\alpha)(c\alpha)(e\alpha)^2\,d_x, \end{aligned}$$

and one invariant of the cubic,

$$I = (ab)^2(cd)^2(ac)(bd).$$

The degrees of these concomitants in the coefficients of a_x^3 are respectively 1, 3, 3, 5; 4. Referring to formula (9) we see that homogeneity admits only the covariants L^{II} and L^{III}, of degree 3 in those coefficients. There are to be determined λ_2 and λ_3 in the expression

$$A_x^4 = I.\alpha_x^4 + (\lambda_2 L^{\mathrm{II}} + \lambda_3 L^{\mathrm{III}})\,d_x^3.$$

In this expression substitute for α_x^4 the conjunctive $\alpha_x^4 + u_x.e_x^3$, and require the terms involving parameters u to disappear identically.

$$0 = I.d_x^3\,u_x + (\tfrac{5}{8}\lambda_2 I.\,u_x - \tfrac{3}{8}\lambda_3 I.u_x)\,d_x^3,$$
$$\therefore \quad 8 + 5\lambda_2 - 3\lambda_3 = 0. \tag{10}$$

Of this equation there are two linearly independent solutions; as the simplest, we adopt these:

$$\begin{aligned} &1)\quad \lambda_2 = -\tfrac{8}{5},\ \lambda_3 = 0; \\ &2)\quad \lambda_2 = \quad 0,\ \lambda_3 = \tfrac{8}{3}. \end{aligned}$$

There are thus seen to be two linearly independent semi-combinant groundforms A_x^4, $A_x'^4$, and a simple infinity of combinations $A_x^4 + m.A_x'^4$;

$$A_x^4 = \alpha_x^4 - \tfrac{8}{5}\,\frac{(ab)^2(ac)(b\alpha)(c\alpha)^2\,\alpha_x}{(ab)^2(ac)(bd)(cd)^2}\,.\,d_x^3, \tag{11}$$

$$A_x'^4 = \alpha_x^4 + \tfrac{8}{3}\,\frac{(a\alpha)^3(bc)^2(b\alpha)\,c_x}{(ab)^2(ac)(bd)(cd)^2}\,.\,d_x^3. \tag{12}$$

As a second example, let it be required to find all semi-combinant groundforms in the simultaneous system of a ternary quartic and cubic, denoted by α_x^4 and a_x^3 respectively. For the ternary cubic a complete reduced system is given by Gordan.* Our covariants L are to be of order 1, and contain symbols α to the degree 4. The symbolic expression of an L may contain either α_x and three α's in determinant-factors; or an a_x, with four α's in determinant-factors. There correspond to these, in Gordan's tables, mixed concomitants (Zwischenformen) either of class 3, order 0, or of class 4, order 1. Of the former we find two, of the latter three. Their degrees in the coefficients of a_x^3 are respectively 3, 5, 5, 7, 8. There are two invariants, of degrees 4 and 6. With the aid of these there can be formed homogeneous expressions involving the first four covariants L, but none containing that of degree 8.

The symbolic expressions are

1) of the invariants,

$$S = (abc)(abd)(acd)(bcd),$$
$$T = (abc)(abd)(ace)(bcf)(def)^2;$$

2) of the four covariants L,

$$L^{\mathrm{I}} = (abc)(ab\alpha)(ac\alpha)(bc\alpha)\,\alpha_x,$$
$$L^{\mathrm{II}} = (abc)(abd)(ace)(bc\alpha)(de\alpha)^2\,\alpha_x,$$
$$L^{\mathrm{III}} = (abc)(ab\alpha)(ac\alpha)(bcd)(de\alpha)^2\,e_x,$$
$$L^{\mathrm{IV}} = (abc)(abd)(ace)(bcf)(de\alpha)^2(fg\alpha)^2\,g_x.$$

Instead of determining the most general semi-combinant groundform and singling out four independent ones by particular values given to the parameters, I will for simplicity's sake determine the particular ones, and of them compound the most general.

The four shall be these:

$$S.F_1 = S.\alpha_x^4 + \lambda_1 L^{\mathrm{I}}\ .a_x^3,$$
$$T.F_2 = T.\alpha_x^4 + \lambda_2 L^{\mathrm{II}}\ .a_x^3,$$
$$T.F_3 = T.\alpha_x^4 + \lambda_3 L^{\mathrm{III}}.a_x^3,$$
$$S^2.F_4 = S^2.\alpha_x^4 + \lambda_4 L^{\mathrm{IV}}.a_x^3.$$

Applying the condition that each of these forms F shall vanish identically for

*Ueber ternäre Formen dritten Grades, Math. Ann. I, pp. 101–102.

$\alpha_x^4 = a_x^3 . u_x$, I find, after somewhat laborious reduction of the resulting symbolic products,* the values of the constants

$$\lambda_1 = -2, \ \lambda_2 = -2, \ \lambda_3 = -6, \ \lambda_4 = \qquad ;$$

so that the simple semi-combinants $F_1 \dots . F_4$ have the expressions

$$\left.\begin{aligned} F_1 &= \alpha_x^4 - \ \ 2\frac{L^{\mathrm{I}}}{S} . a_x^3, \\ F_2 &= \alpha_x^4 - \ \ 2\frac{L^{\mathrm{II}}}{T} . a_x^3, \\ F_3 &= \alpha_x^4 - \ \ 6\frac{L^{\mathrm{III}}}{T} . a_x^3, \\ F_4 &= \alpha_x^4 - 36\frac{L^{\mathrm{IV}}}{S^2} . a_x^3 . \end{aligned}\right\} \tag{13}$$

The most general semi-combinant groundform is the linear combination of these four, with arbitrary multipliers,

$$F = l_1 F_1 + l_2 F_2 + l_3 F_3 + l_4 F_4 .$$

From these examples it is sufficiently clear what method must be followed in obtaining all semi-combinant groundforms of the system of two given quantics. The covariants L must first be enumerated. While not every L will give necessarily an independent semi-combinant, yet none gives more than one; and we may formulate the following conclusion:

The number of linearly independent semi-combinant groundforms of two quantics of different order does not exceed the number of linearly independent covariants L, linear in the coefficients of the higher quantic, and of order equal to the difference in orders of the given quantics.

Thus a superior limit is established, which will be confirmed later, when also it will appear why this number falls often below that limit.

§4. *Reduced Form-system of Semi-Combinants derivable from that of General Covariants.*

Any covariant of a system of two quantics α_x^m, a_x^n, which is also a semi-combinant of the system, is unaltered when in its explicit formula the coefficients of

*The most useful reduction-formulæ are given by Clebsch and Gordan, Ueber cubische ternäre Formen, Math. Ann. 6, pp. 448-9 and 467.

a_x^m are replaced by those of $a_x^m + u_x^{m-n} a_x^n$. This is indeed the definition of a semi-combinant. Accordingly, giving the arbitrary parameters in u_x^{m-n} special values, so that $a_x^m + u_x^{m-n} a_x^n$ becomes a semi-combinant groundform A_x^m, we may notice in particular that

All semi-combinant invariants and covariants are unaltered by the substitution

$$a_x^m \sim A_x^m,$$

A_x^m denoting any semi-combinant groundform.

Assume that in every system such semi-combinant groundforms exist. Suppose the aggregate of covariants of the system subjected to the above substitution,

$$a_x^m \sim A_x^m.$$

Every covariant will be transformed into a semi-combinant, for it becomes a covariant of a semi-combinant. Suppose further all identical equations to be explicitly given, which express covariants in terms of the finite number of covariants constituting a reduced form-system. These syzygies being subjected to the same substitution,

$$a_x^m \sim A_x^m,$$

will become syzygies between semi-combinants, reducible covariants becoming reducible semi-combinants, and the reduced form-system of ordinary covariants yielding by transformation a set of semi-combinants in terms of which all other semi-combinants are expressed as rational entire functions. Hence the theorem:

The reduced form-system of semi-combinants cannot contain more irreducible concomitants than that of ordinary covariants.

That it must contain fewer will appear subsequently (see §6); for at least one covariant of the reduced system will vanish identically.

This transformation gives all the semi-combinants of a system, and a part of their relations *inter se*, as soon as all covariants of the system, their syzygies, and a single semi-combinant groundform are known. The discovery of the latter is therefore the essential step in proceeding from a theory of ordinary invariants, such as already exists in greater or less completeness, to a correspondingly complete theory of the subgroup, semi-combinants. Particular examples like those of §3 do not suffice. We must address ourselves to the general question, how to find semi-combinant groundforms in the system of two or more given quantics.

§5. *Affiliated Forms defined and determined as Covariants in a given System.*

A class of concomitants that are certainly semi-combinant groundforms of two given quantics α_x^m and a_x^n, $(m > n)$, are those satisfying identically linear differential equations, invariant *per se*, having as coefficients functions of the coefficients of the quantic a_x^n; provided such solutions are completely determinate. (Otherwise not the particular solution, but the aggregate of all solutions, would be semi-combinant.) The defining equations must equal in number the arbitrary parameters in the conjunctive $\alpha_x^m + u_x^{m-n} a_x^n$. They may comprise that number of invariants, each equated to zero, or any part of the number may be represented by a covariant equated identically to zero—an *invariantive set* of differential equations. The solutions will be rational covariants if the defining equations are all linear; were any of higher degree the solutions would still be semi-combinants, but irrational. I intend here to examine solutions of sets of equations found by requiring a covariant of order $m - n$ to vanish identically. I shall speak of these equations in the aggregate as a *characteristic equation*, and call their solution an *affiliated form* or *affiliant* of the system of α_x^m and a_x^n.

An affiliated form or affiliant of the system of α_x^m *and* a_x^n *is any covariant of the type* $A_x^m = \alpha_x^m + l_x^{m-n} a_x^n$, *which satisfies a characteristic equation obtained by equating identically to zero a covariant of order* $m - n$, *linear in the coefficients of* A_x^m.

If the quantic of lower order be a quadric, there is of course only one affiliated form,* that one, namely, which is apolar or conjugate to the quadric. Two examples of this simple case will make the above definition more intelligible.

First example: System of binary cubic and quadric.

Required the values of l_1 and l_2 which render $\alpha_x^3 + l_x . a_x^2 = A_x^3$ an affiliated form. There are two covariants of A_x^3 and a_x^2 which can give equations of definition,† but the identical vanishing of either one involves the vanishing of the other. The one of lowest degree is this:

$$(aA)^2 A_x \equiv 0, \tag{14}$$

or expanding,

$$3\,(a\alpha)^2 \alpha_x + (ab)^2 l_x + 2\,(ab)(al)\, b_x \equiv 0,$$

* For the condition of apolarity involves every other covariant condition of the prescribed order.
† Named p and q by Gordan-Kerschensteiner, p. 323.

and reducing,

$$3\,(a\alpha)^2\alpha_x + 2\,(ab)^2 l_x \equiv 0.$$

Since the only term involving the parameters l_1, l_2 contains the factor l_x, it is only necessary to solve for this linear form

$$l_x = -\tfrac{3}{2}.\frac{(a\alpha)^2\,\alpha_x}{(ab)^2}; \tag{15}$$

hence the required affiliated form is

$$A_x^3 = \alpha_x^3 - \tfrac{3}{2}.\frac{(a\alpha)^2\,\alpha_x.b_x^2}{(ab)^2}. \tag{15a}$$

Second example: System of binary quartic and quadric.

Of the six quadratic covariants of a quartic and a quadric, two are linear in the coefficients of the quartic, those named ψ and Ψ by Clebsch (*Binäre Formen*, pp. 213, 214). If ψ become identically zero, so also will Ψ. The two give thus only one characteristic equation for an affiliated form: $A_x^4 = \alpha_x^4 + l_x^2.a_x^2$, namely,

$$(aA)^2 A_x^2 \equiv 0, \tag{16}$$

i. e. $$6\,(a\alpha)^2\,\alpha_x^2 + (ab)^2\,l_x^2 + 4\,(ab)(al)\,b_x\,l_x + (al)^2\,b_x^2 \equiv 0;$$

or, after reducing one term,

$$6\,(a\alpha)^2\,\alpha_x^2 + 3\,(ab)^2\,l_x^2 + (al)^2\,b_x^2 \equiv 0.$$

This is equivalent to three equations for parameters, and these are linearly independent, for otherwise we find that a multiple of the discriminant of the quadric must vanish.

Instead of solving these three equations, compounding the resultant determinant-quotients with proper multipliers into the form l_x^2, and reducing its expression to covariant form, I find it more convenient to solve by a convergent series. The equation for l_x^2 is, calling $(ab)^2 = \Delta$,

$$l_x^2 = -\,2\,\frac{(a\alpha)^2\,\alpha_x^2}{\Delta} - \tfrac{1}{3}\,b_x^2.\frac{(al)^2}{\Delta}.$$

By repeated substitution of this expression for l_x^2 in the second member—in itself—we have

$$l_x^2 = -\frac{2\,(a\alpha)^2\,\alpha_x^2}{\Delta} + \tfrac{2}{3}\,\frac{(a\alpha)^2(b\alpha)^2\,c_x^2}{\Delta^2} - \tfrac{2}{9}\,\frac{\Delta.(a\alpha)^2(b\alpha)^2\,c_x^2}{\Delta^3} + \text{etc.},$$

$$l_x^2 = -\,2\,\frac{(a\alpha)^2\,\alpha_x^2}{\Delta} + \frac{(a\alpha)^2(b\alpha)^2\,c_x^2}{\Delta^2}\left(\tfrac{2}{3} - \tfrac{2}{9} + \tfrac{2}{27} - \dots\right)$$

$$= -\,2\,\frac{(a\alpha)^2\,\alpha_x^2}{\Delta} + \tfrac{1}{2}.\frac{(a\alpha)^2(b\alpha)^2\,c_x^2}{\Delta^2}. \tag{17}$$

Therefore the affiliated quartic of the system is

$$A_x^4 = \alpha_x^4 - 2\,\frac{(a\alpha)^2\,\alpha_x^2}{\Delta}\,.\,b_x^2 + \tfrac{1}{2}\,.\,\frac{(a\alpha)^2.(b\alpha)^2}{\Delta^2}\,.\,c_x^2\,.\,d_x^2\,. \tag{17a}$$

These examples exhibit one mode of finding the affiliated form when its characteristic equation is given. Such a solution must be a semi-combinant groundform, for it is a covariant by virtue of the covariant defining equation; it is of the type $\alpha_x^m + l_x^{m-n}\,a_x^n$; and as it is uniquely determined from the conditions, it must be unaltered by the substitution

$$\alpha_x^m \sim \alpha_x^m + u_x^{m-n} a_x^n,$$

for that affects neither its type nor its characteristic equation.

This last remark may be stated more clearly by employing a functional notation for covariants, and by this means an important property of the covariant l_x^{m-n} or L may be developed. Designate by α and a the quantics α_x^m and a_x^n; by U any arbitrary quantic u_x^{m-n}; by $G(\alpha)$ and $L(\alpha)$ two covariants, of order $m-n$, of α_x^m and a_x^n, linear in the coefficients of the former, such that $L(\alpha)$ satisfies the characteristic equation

$$G\big(\alpha + aL(\alpha)\big) \equiv 0. \tag{18}$$

On account of the linear structure of these covariants we have

$$\left.\begin{aligned} G\big(\alpha + aL(\alpha)\big) &= G(\alpha) + G\big(a.L(\alpha)\big), \\ L(\alpha + a.U) &= L(\alpha) + L(aU). \end{aligned}\right\} \tag{19}$$

If for α_x^m we substitute $\alpha_x^m + u_x^{m-n} a_x^n$, we have from the identical equation (18),

$$G\big(\alpha + aU + aL(\alpha + aU)\big) \equiv 0\,;$$

and therefore, since L is uniquely determined by (18), the identity

$$\alpha + aL(\alpha) \equiv (\alpha + aU) + aL(\alpha + aU), \tag{20}$$

showing that the affiliant $\alpha + aL(\alpha)$ is unaltered by the substitution $\alpha_x^m \sim \alpha_x^m + u_x^{m-n} a_x^n$; or, in other words, *every affiliant is a semi-combinant groundform of the system.*

Equation (20) expanded by (19) is this:

$$\alpha + aL(\alpha) \equiv \alpha + aU + aL(\alpha) + aL(a.U),$$

or, on dropping like terms,

$$aU + aL(aU) \equiv 0,$$

and without the factor a,

$$L(aU) \equiv -\,U. \tag{21}$$

Remembering that L is determined as the quotient of an integral rational simultaneous covariant of α and a by an invariant of a alone, we may set

$$L(\alpha) = \frac{K(a,\,\alpha)}{I(a)}\,,$$

whereby equation (21) becomes

$$K(a,\, aU) + U.I(a) \equiv 0. \tag{21a}$$

This may be regarded as a defining equation for $K(a,\,\alpha)$, in accordance with which it is to be derived, linear in the coefficients of α, from a suitable given invariant $I(a)$. The solution of this problem would lead directly to the formation of a semi-combinant groundform.

In a given system, how many affiliants can occur? Certainly not more than the number of independent characteristic equations; that is, than the number of independent covariants G, G', G'', ... of order $m-n$, linear in the coefficients of the higher quantic. Hence, as was seen above in the closing theorem of §3, *the number of independent affiliants has for an upper limit the limit of the number of semi-combinant groundforms in the same system.*

Further, there is an obvious reason why not all linearly independent covariants G, G', G'', etc., need furnish different affiliants when they are employed in characteristic equations. It may happen that one or more of the remaining G's can be represented as a simultaneous covariant of the first and of a_x^n. In that case, of course, the solution of $G=0$ would satisfy $G'=0$ and the rest belonging to the system of G and a. *This case always occurs* if the characteristic equation has a determinate solution whose covariant L is not a mere multiple of G.

If we suppose, as in (18),

$$G\big(\alpha + aL(\alpha)\big) = 0,$$

then we have, according to (21),

$$L(aU) = -U, \text{ and } L\big(aL(\alpha)\big) = -L(\alpha).$$

Therefore the solution L_1 of the equation

$$L\big(\alpha + aL_1(\alpha)\big) \equiv 0$$

is L itself. For we have

$$L\big(\alpha + aL(\alpha)\big) \equiv L(\alpha) + L\big(aL(\alpha)\big) \equiv L(\alpha) - L(\alpha) \equiv 0, \tag{22}$$

and this we will adopt as the normal form of the characteristic equation of an affiliant.

Here, of course, $L(\alpha)$ may be radically different from $G(\alpha)$, or it may differ only by a factor. In either case, $L(\alpha)$ must be a covariant of $G(\alpha)$; for by virtue of (18) and (22), if $L(A) \equiv 0$, it follows that $G(A) \equiv 0$, and conversely.

We can name not only a superior limit, but probably also one inferior limit to this number of independent affiliants. *This number is not less than the number of irreducible invariants of the lower quantic a_x^n, diminished by the number of syzygies of the first kind.* From such invariants there can be derived covariants $G(\alpha)$ by an Aronhold process, which shall replace coefficients of a_x^n by those of the polar, $\alpha_x^n \alpha_y^{m-n}$. The latter will be independent save as derivatives of the syzygies shall indicate relations. Such covariants G will not vanish simultaneously except by virtue of special invariant properties of the lower quantic. There is lacking a proof that each such G has corresponding to it a determinate solution L of the equation

$$G(\alpha + aL(\alpha)) \equiv 0.$$

As I have no complete proof of this, it may be left for the present to be investigated in each separate case discussed.*

A question of some interest may be mentioned here. Suppose two affiliants determined by characteristic equations, and the latter put into normal form (*vid.* (22)), so that we may write

$$L_1(\alpha + aL_1(\alpha)) \equiv 0,$$
$$L_2(\alpha + aL_2(\alpha)) \equiv 0.$$

If now these equations be combined linearly, how will the solution depend upon the solutions L_1 and L_2 of the separate equations? If, namely,

$$G_3(\alpha + aL_3(\alpha)) = (\lambda_1 . L_1 + \lambda_2 . L_2)(\alpha + aL_3(\alpha)) \equiv 0, \tag{23}$$

what relation subsists between L_1, L_2, L_3?

Expanding (23) by (19), and using (21), we find

$$\lambda_1 . L_1(\alpha) + \lambda_2 . L_2(\alpha) - \lambda_1 . L_3(\alpha) - \lambda_2 . L_3(\alpha) \equiv 0,$$
$$L_3(\alpha) = \frac{\lambda_1 . L_1(\alpha) + \lambda_2 . L_2(\alpha)}{\lambda_1 + \lambda_2}. \tag{24}$$

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* The most obvious point of attack for this problem is offered by equation (21a).

The same considerations show that for any number of equations, $G_i(\alpha + aL_i) \equiv 0$, with determinate solutions L_i,

$$\Sigma\lambda_i L_i\left(\alpha + \frac{a.\Sigma\lambda_i L_i(\alpha)}{\Sigma\lambda_i}\right) \equiv 0. \tag{24a}$$

Loosely stated, this would show that the solution of $\Sigma\lambda_i L_i \equiv 0$ is $\frac{\Sigma\lambda_i L_i}{\Sigma\lambda_i}$.

The linear aggregate of characteristic equations is projectively related to the aggregate of corresponding solutions, and is perspectively related to them, if the characteristic equations are in normal form.

For the solution of (24a) is identically

$$\frac{\Sigma\lambda_i.\alpha + a\Sigma\lambda_i L_i(\alpha)}{\Sigma\lambda_i},$$

and this is

$$\frac{\Sigma\lambda_i(\alpha + aL_i(\alpha))}{\Sigma\lambda_i},$$

wherein every term of the numerator $(\alpha + aL_i(\alpha))$ is the solution of the corresponding characteristic equation

$$L_i \equiv 0.$$

This shows how a given equation may have its solution indeterminate. If in (24a) we assume $\Sigma\lambda_i = 0$, the solution would become

$$\Sigma\lambda_i\alpha + a\Sigma\lambda_i L_i(\alpha) = a.\Sigma\lambda_i L_i(\alpha).$$

Since this is necessarily a semi-combinant, the solution of characteristic equation

$$\Sigma\lambda_i L_i(\alpha + aU) \equiv 0$$

is indeterminate. It is namely the quantic a_x^n multiplied by an arbitrary quantic U of order $m - n$; for we have by (21)

$$\Sigma\lambda_i L_i(aU) \equiv -\Sigma\lambda_i.U \equiv U.\Sigma\lambda_i \equiv 0.$$

Equations of this sort may be regarded as forming a singular system for defining the quantic a_x^n, which is indeed a semi-combinant by virtue of being a covariant and unaltered by the substitution

$$\alpha_x^m \sim \alpha_x^m + u_x^{m-n}.a_x^n.$$

Every linear combination of normal characteristic equations has a determinate solution except when the combination $\Sigma\lambda_i L_i \equiv 0$ *is one of the singular system having* $\Sigma\lambda_i = 0$. *Of any singular characteristic equation the solution is indeterminate, being the quantic of lower order multiplied by an arbitrary function.*

§6. *Every Semi-Combinant Groundform is an Affiliant. Production of its Characteristic Equation.*

The inquiry for a single semi-combinant groundform in the system of two quantics α_x^m, a_x^n (or α and a) resulted in the finding of a whole class of such groundforms, the affiliants of §5. Every affiliant is a semi-combinant groundform, and as such suffices for the determination of all covariants that share its semi-combinant property. It is now to be shown that, conversely, every semi-combinant groundform is an affiliant.

Let the rational covariant of α_x^m and a_x^n:

$$\left.\begin{aligned} A_x^m &\equiv \alpha_x^m + l_x^{m-n} a_x^n, \\ \text{or for brevity}\qquad A &\equiv \alpha \;\; + a.L(\alpha), \end{aligned}\right\} \tag{25}$$

be a semi-combinant; i. e. let it be unaltered by the substitution

$$\alpha_x^m \sim \alpha_x^m + u_x^{m-n} a_x^n.$$

Then the N_m equations involved in the identity or definition (25) may be regarded as equations of the first degree in coefficients of α, by whose aid those coefficients are to be expressed as rational linear functions of coefficients of A. But the quantic α is by hypothesis not determinate when A is known; on the contrary, it contains implicitly the N_{m-n} parameters of u_x^{m-n}. Therefore all determinants of a matrix having N_m rows and $(N_m - N_{m-n})$ columns—*linear* in coefficients of A, but not usually so in those of a—must vanish. These several zero-determinants can be multiplied with suitable power products of variables $(x_1, x_2, x_3, \ldots . x_r)$, so that their sum equated to zero will constitute the covariant characteristic equation described in §5, by virtue of which the quantic A is an affiliant. This process of proof is, however, unnecessarily expanded, and can be compressed into the following symbolic form, a reproduction of (22).

Since A is a semi-combinant,

$$\begin{aligned} \alpha + a.U + aL(\alpha + aU) &\equiv \alpha + aL(\alpha); \\ \text{therefore}\qquad aU + aL(aU) &\equiv 0, \\ L(aU) &\equiv - U, \end{aligned}$$

U denoting an arbitrary quantic $U = u_x^{m-n}$. This property of the covariant $L(\alpha)$

depends only upon the order $m-n$ of the quantic U. Hence for U we may write L itself:

$$L(aL(\alpha)) \equiv -L(\alpha),$$

or

$$L(\alpha)+L(aL(\alpha)) \equiv L(\alpha+aL(\alpha)) \equiv 0,$$

and this is exactly the characteristic equation of an affiliant in its normal form (*vid.* (22)). This theorem may be formulated as follows:

If a semi-combinant groundform be written in symbols of two quantics α and a

$$A=\alpha+a.L(\alpha),$$

where $L(\alpha)$ is a rational covariant, linear in the coefficients of α, then this semi-combinant groundform is an affiliant of a, satisfying the equation

$$L(A) \equiv 0.$$

As L is always fractional, it remains only to multiply it by a suitable invariant of a to make it an entire covariant of A and a.

§7. *Further Examples of Affiliants.*

In accordance with the foregoing, we can write immediately the characteristic equations of the semi-combinants calculated in §3. Of the binary quartic and cubic, the semi-combinant A_x^4 satisfies the defining equation (see (11))

$$(ab)^2(ac)(bA)(cA)^2A_x \equiv 0.$$

This is obviously derived from the discriminant of the cubic

$$(ab)^2(ac)(bd)(cd)^2.$$

The other semi-combinant groundform $A_x'^4$ (*vid.* (12)) has for its equation

$$(aA)^3(bc)^2(bA)\,c_x \equiv 0.$$

This can be simplified by equating to zero its transvectant with the Hessian of the cubic: $(de)^2 d_x e_x$. This gives

$$(aA)^3(bc)^2(bA)(cd)(de)^2 e_x \equiv 0.$$

One reduction brings this into the form

$$(aA)^3 A_x.(bc)^2(be)(cd)(de)^2 \equiv 0,$$
$$\therefore\ (aA)^3 A_x \equiv 0,$$

if the cubic a_x^3 has no square factor. This is evidently a derivative of the iden-

tically vanishing invariant $(ab)^3$. Just as the two linear covariants $L'''_{(a)}$ and $L'_{(a)}$ are found to give characteristic equations having the same solution $A'\ (= A'^4_x)$,

$$L'(A') \equiv 0, \quad L'''(A') \equiv 0,$$

so the solution $A\ (= A^4_x$, formula (11)) of $L''(A) \equiv 0$ satisfies the fourth, the only remaining independent linear covariant equation $L^{\text{IV}}(A) \equiv 0$. For L^{IV} is the transvectant of L'' with the Hessian of the cubic.

In the second example, the system of the ternary quartic and cubic, the characteristic equations of F_1, F_2, F_3, F_4 can be written down directly from the symbolic expressions of L^{I}, L^{II}, L^{III}, L^{IV}. Our special interest will be directed toward the fifth linear covariant $L^{\text{V}}(\alpha)$ of degree 8 in the coefficients of the cubic. By reason of its degree alone it was found impossible to have a corresponding semi-combinant groundform, since there is no invariant of degree 9. But if we use this covariant in a characteristic equation for an affiliant, must it not disclose a corresponding semi-combinant? Making trial, I learn that $L^{\text{V}}(\alpha)$ *is itself a semi-combinant, and hence cannot determine an affiliant.* The proof is condensed by the use of Gordan's notation and reductions as follows:

$$L^{\text{V}}(\alpha) = \alpha_t^2 \alpha_s^2 (stx), \tag{26}$$

where u_t^3 and u_s^3 are fundamental contravariants,

$$\left.\begin{aligned} u_s^3 &= (abc)(abu)(acu)(bcu), \\ u_t^3 &= a_s b_s u_s (abu)^2. \end{aligned}\right\} \tag{26a}$$

To prove $L^{\text{V}}(\alpha)$ a semi-combinant we must show that

$$L^{\text{V}}(\alpha + Ua) \equiv L^{\text{V}}(\alpha)\,; \text{ i. e. } L^{\text{V}}(Ua) \equiv 0.$$

Since U and a denote u_x and a_x^3, we have

$$\begin{aligned} 2L^{\text{V}}(Ua) &= u_t a_t a_s^2 (stx) + a_t^2 a_s u_s (stx) \\ &= \tfrac{1}{3} S.(ttx)\, u_t{}^{*} + \tfrac{1}{3} T.(xss)\, u_s{}^{\dagger} \equiv 0, \end{aligned}$$

both parts vanishing identically.

When the two fundamental quantics are of orders differing by one, and the lower order is even, one affiliant can be found very simply, whatever the number

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* Gordan, Ueber ternäre Formen dritten Grades, Math. Ann. I, p. 104, Tafel IV, 2.

† Ibidem, p. 108, Tafel XII, 5. Cf. Math. Ann. VI, p. 467.

of variables in the two quantics. Assuming two quaternary quantics of orders $2n$ and $2n+1$,

$$\alpha_x^{2n+1} \text{ and } a_x^{2n},$$

I mean the affiliant
$$\alpha_x^{2n+1} + l_x . a_x^{2n} \equiv A_x^{2n+1},$$

which satisfies the characteristic equation

$$(abcA)^{2n} A_x \equiv 0. \tag{27}$$

Expanding this equation,

$$\begin{aligned} 0 &\equiv (abc\alpha)^{2n}\alpha_x + \frac{1}{2n+1}(abcd)^{2n} l_x + \frac{2n}{2n+1}(abcd)^{2n-1}(abcl)\, d_x \\ &\equiv (abc\alpha)^{2n}\alpha_x + \frac{n+2}{4n+2}(abcd)^{2n} l_x ; \end{aligned}$$

$$\therefore\ l_x = -\frac{4n+2}{n+2} \cdot \frac{(abc\alpha)^{2n}\alpha_x}{(abcd)^{2n}},$$

$$A_x^{2n+1} = \alpha_x^{2n+1} - \frac{4n+2}{n+2} \cdot \frac{(abc\alpha)^{2n}\alpha_x}{(abcd)^{2n}} . d_x^{2n}.$$

For quantics in r variables, the same equation will give

$$A_x^{2n+1} = \alpha_x^{2n+1} - \frac{r(2n+1)}{2n+r} \cdot \frac{(ab \ldots\ldots \alpha)^{2n}\alpha_x}{(ab \ldots\ldots d)^{2n}} . d_x^{2n}. \tag{28}$$

Two binary quantics, the lower of odd order, a_x^{2n+1}, the other of order higher by unity, α_x^{2n+2}, have an affiliant determined almost as simply as the above. Set $A_x^{2n+2} = \alpha_x^{2n+2} + l_x a_x^{2n+1}$, and take as characteristic equation

$$(aA)^{2n+1} A_x \equiv 0. \tag{29}$$

This is, when expanded,

$$(a\alpha)^{2n+1}\alpha_x + \frac{1}{2n+2}(ab)^{2n+1} l_x + \frac{2n+1}{2n+2}(ab)^{2n}(al)\, b_x \equiv 0.$$

Since the second term contains the vanishing invariant $(ab)^{2n+1}$, the equation becomes

$$(a\alpha)^{2n+1}\alpha_x + \frac{2n+1}{2n+2}(ab)^{2n}(al)\, b_x \equiv 0.$$

For convenience, denote by p_x the left-hand member of this equation, and by k_x^2

the covariant $(cd)^{2n}c_x d_x$. Since $p_x \equiv 0$, we may equate to zero its transvectant,

$$0 \equiv (pk)\,k_x \equiv (cd)^2\ (a\alpha)^{2n+1}(\alpha c)\,dx + \frac{2n+1}{2n+2}\,(ab)^{2n}(cd)^{2n}(al)(bd)\,c_x$$

$$\equiv (cd)^{2n}(a\alpha)^{2n+1}(\alpha c)\,d_x + \frac{2n+1}{4n+4}\,(ab)^{2n}(cd)^{2n}(ac)(bd)\,.\,l_x\,,$$

$$l_x = +\,\frac{4\,(n+1)}{2n+1}\cdot\frac{(ab)^{2n}(ca)^{2n+1}(a\alpha)\,b_x}{(ab)^{2n}(cd)^{2n}(ac)(bd)}\,,$$

$$A_x^{2n+2} = \alpha_x^{2n+2} + \frac{4\,(n+1)}{2n+1}\cdot\frac{(ab)^{2n}(ca)^{2n+1}(a\alpha)\,b_x}{(ab)^{2n}(cd)^{2n}(ac)(bd)}\,.\,d_x^{2n+1}. \tag{30}$$

Whatever the number of variables, if the orders of the two quantics differ only by unity, the *discriminant* of the lower quantic furnishes by evectation a characteristic equation whose solution can be obtained immediately as in (27) and (28). *Thus there is proved in general the existence of an affiliant when the given forms are of order n and $n+1$.* Where the difference in orders is two or more, the existence of affiliants is yet to be confirmed.

As showing interesting processes capable of extension, consider a binary quartic and sextic, a_x^4 and α_x^6. Let it be required to determine the affiliant

$$A_x^6 = \alpha_x^6 + l_x^2\,a_x^4$$

from the equation $\qquad (aA)^4 A_x^2 \equiv 0\,,$

or l_x^2 from the reduced equation

$$(a\alpha)^4\alpha_x^2 + \tfrac{1}{3}\,(ab)^4 l_x^2 + \tfrac{2}{5}\,(ab)^2(al)^2\,b_x^2 \equiv 0\,.$$

Transposing, this gives

$$l_x^2 = -\,3\,\frac{(a\alpha)^4\alpha_x^2}{(ab)^4} - \tfrac{6}{5}\,\frac{(la)^2(ab)^2\,b_x^2}{(ab)^4}\,. \tag{31}$$

The scheme for solving this equation without leaving the domain of rational covariants is now to substitute the value of l_x^2, given in this second member, into the second term of that second member, then to repeat the process *ad infinitum.* This is admissible if the resulting series shall be found to converge. For the convenient execution of this plan I will designate by P the operation which from any quadric u_x^2 derives $(ua)^2 a_x^2$. Then for the second term of the second member of (31) I have

$$-\,\frac{6}{5}\,\frac{(la)^2(ab)^2\,b_x^2}{(ab)^4} = r\,.\,P^2\,(l_x^2),$$

where

$$i = (ab)^4,\quad r = -\,\frac{6}{5i}\,.$$

Calling the first term F,

$$F = -\frac{3}{i}(a\alpha)^4 \alpha_x^2,$$

equation (31) gives the series

$$\begin{aligned} l_x^2 &= F + rP^2(F) + r^2P^4(F) + r^3P^6(F) + \ldots . \\ &= (1 + rP^2 + r^2P^4 + r^3P^6 + \ldots .)(F). \end{aligned} \tag{32}$$

The problem is reduced to the summation of the infinite series of operations upon a quadric F,

$$C(F) = (1 + rP^2 + r^2P^4 + \ldots .)(F). \tag{33}$$

There is a reduction-formula given by Clebsch* which is, when adapted for even powers of the operator P: $(j = (ab)^2(bc)^2(ca)^2)$,

$$P^k = i.P^{k-2} - \frac{i^2}{4}P^{k-4} + \frac{j^3}{9}P^{k-6}.$$

Applying this in the series C after the third term, I have

$$C\left(1 + \frac{6}{5} + \frac{9}{25}\,\frac{24j^2}{125i^3}\right) = \left(1 + \frac{6}{5} + \frac{9}{25}\right) - P^2\left(\frac{6}{5i} + \frac{36}{25i}\right) + \frac{36}{25i^2}P^4.$$

Since P^4 can be reduced,

$$P^4 = \frac{i}{2}P^2 + \frac{j}{3}P,$$

this gives the sum,

$$C = \frac{40i^3}{40i^3 + 3j^2} + \frac{15ij}{80i^3 + 6j^2}P - \frac{30i^2}{40i^3 + 3j^2}P^2.$$

Substituting this in (33) and (32), together with the value of F, there results

$$l_x^2 = -\frac{15}{40i^3 + 3j^2}\left\{8i^2 + \tfrac{3}{2}jP - 6iP^2\right\}\left((a\alpha)^4\alpha_x^2\right),$$

or

$$l_x^2 = -\frac{15(a\alpha)^4}{40i^3 + 3j^2}\left(8i^2.\alpha_x^2 + \tfrac{3}{2}j.(b\alpha)^2 b_x^2 - 6i.(b\alpha)^2(bc)^2 c_x^2\right). \tag{35}$$

This completes the determination of the affiliant

$$A_x^6 = \alpha_x^6 + l_x^2.\alpha_x^4.$$

A somewhat simpler affiliant exists, a linear combination of the one just found with that whose characteristic equation is derived from the invariant j,

$$(ab)^2(aA')^2(bA')^2 A_x'^2 \equiv 0.$$

*Clebsch, Binäre Formen, p. 219, (2).

The simplest affiliant should be used in discussing the reduced form-system of semi-combinants. For that purpose, however, the simplest would probably be *not that whose symbolic expression is of simplest type, but that whose characteristic equation is of simplest type.*

This last example gives a form of solution which, if applicable in any particular case, is certainly more elegant than the separation of the characteristic equation into its constituent parts, the determination of the auxiliary quantic l_x^{m-n} from these parts as a determinant quotient, and the reduction of this quotient to simpler covariants. The longer process, however, gives occasion to investigate directly whether the solution is determinate. For any one case such an investigation can be made to depend upon a canonical form of the quantic of lower order, if a canonical form is known. An interesting alternative is offered by the series $C(F)$, formula (33). In general, P would be replaced by some covariant process which preserves (as P does) the order of the operand in the variables and its degree in the coefficients, say by the process Π. The generalized problem would be to find under what conditions the series

$$(1 + r\Pi + r^2\Pi^2 + \ldots \text{ in inf.})(F)$$

represents a uniformly convergent series, where r denotes some invariant factor.

Only one such series fully treated has met my eye, that referred to above in Clebsch's treatment of the system of a binary quartic and quadric. By the aid of his results it is possible to find affiliants of any order n to the binary quartic a_x^4, which shall satisfy the equation

$$(aA)^4 A_x^{n-4} \equiv 0,$$

or the equation

$$(ab)^2 (aA)^2 (bA)^2 A_x^{n-4} \equiv 0.$$

I should mention, of course, the system of a quadric and an n^{ic}, where the summation of a geometric series is sufficient for calculating an affiliant or apolar quantic.

§8. *Applicability to Normal-form Problem, with a Special Theorem.*

In geometric research, two quantics equated to zero denote two algebraic loci. The relative positions of these two loci may be investigated projectively, when the aggregate of all simultaneous covariants must be discussed. If, however, not the two loci but their intersection is to be investigated projectively,

then the sub-group, their semi-combinants, is alone to be discussed. For the intersection of two loci,

$$a_x^m = 0, \; a_x^n = 0, \qquad (m > n),$$

is equally the intersection of the loci

$$a_x^m + u_x^{m-n} a_x^n = 0, \; a_x^n = 0;$$

that is, of the lower-order locus with the conjunctive of the two. Projective covariants of the intersection-locus are therefore covariants of the conjunctive and of the lower-order locus, independent of the arbitrary coefficients of u_x^{m-n}; that is to say, they are semi-combinants of a_x^m and a_x^n. Ordinarily speaking, an intersection-locus is a geometric form when the two quantics contain at least three variables. Thus semi-combinants of two ternary quantics belong peculiarly to the complete system of points in which two algebraic curves intersect. More interesting is the consideration that projective properties of such twisted curves as are complete intersections of two algebraic surfaces ("elementary curves") will be represented by relations among semi-combinants of the intersecting surfaces (i. e. of their associated quantics).* For point-systems in three dimensions, or for curves in more than three dimensions, systems of at least three quantics and their semi-combinants will enter the discussion. If, further, the orders of two intersecting loci be equal, then not semi-combinants but complete combinants (doubly semi-combinants) of these two quantics must be considered.

A special problem arises in the theory of Abelian integrals on "elementary" twisted curves, i. e. on the non-singular intersection-curve of two algebraic surfaces, usually of unequal order. Normal-forms for integrals of the different species are to be fixed by algebraic considerations. For the reasons above specified, a normal-form ought obviously to be a semi-combinant of the two surfaces. For this as well as for other applications, the following theorem is valuable:

Theorem: If a covariant of two quantics of unlike order, $F(a_x^m, a_x^n)$ or $F(\alpha, a)$ has been found which changes only by multiples of a_x^m and a_x^n when subjected to the substitution

$$a_x^m \sim a_x^m + u_x^{m-n} a_x^n,$$

or

$$\alpha \sim \alpha + Ua,$$

* Semi-combinants of the two surfaces would be, for example, the tact-invariant, the equation of the developable having the intersection-curve for its edge of regression, and the equation of the locus of triple-secants of the intersection-curve.

then a semi-combinant of α *and* a *can be derived from* $F(\alpha, a)$ *by the addition of covariant multiples of* α_x^m *and* a_x^n,

$$F'(\alpha, a) = F'(\alpha + Ua, a) = F(\alpha, a) + M.\alpha + N.a,$$

where M *and* N *denote covariants.*

In proof of this I assume, what is not yet fully demonstrated, that there exists an affiliant groundform $A = \alpha + L.a$. In use, this must be tested until it shall be generally proven. Substituting A for α gives a semi-combinant

$$F'(\alpha, a) = F(A, a) = F(\alpha + a.L, a).$$

But by hypothesis

$$F(\alpha + a.L, a) = F(\alpha, a) + \text{multiples of } \alpha \text{ and } a,$$

(say)

$$= F(\alpha, a) + M.\alpha + N.a;$$

that is to say, *the semi-combinant* $F'(\alpha, a)$ *differs from* $F(\alpha, a)$ *by covariant multiples of* α *and* a, *q. e. d.*

For precision in any proposed application the assumption spoken of must be verified for that case; then the domain of rationality of the covariants F, M and N must be specified and the theorem restated.

§9. *Affiliants and Semi-combinants in a System of more than two Quantics, and in Systems of Mixed Forms.*

In a system of more than two independent quantics of different orders in a given number of variables, those invariant concomitants are semi-combinants of the system which are semi-combinants of every pair of quantics taken separately. These must all be obtainable as concomitants of a derived system of ground-quantics, where each of the given quantics, except the lowest, has been replaced by an affiliant of all quantics of lower order. If, for example, three quantics: f_l, f_m, f_n are the groundforms of the system $(l > m > n)$, then these are to be replaced by

$$f_n = f_n,$$
$$F_m = f_m + f_n.L_{m,n}(f_m),$$
$$F_l = f_l + F_m L_{l,m}(f_l) + f_n L_{l,n}(f_l),$$

F_m being an affiliant of f_n; F_l, a simultaneous affiliant of f_n and F_m. Then it is evident, as in the case of semi-combinants of two quantics, that

Every simultaneous covariant of the three quantics f_n, F_m, F_l *is a semi-combinant of the quantics* f_n, f_m, f_l; *and conversely, that every semi-combinant of the latter set is a simultaneous covariant of the former set.*

The proof that semi-combinant groundforms are affiliants of the several quantics of lower order, and the discussion of the complete system of differential equations satisfied by semi-combinants, appear more interesting even than the corresponding parts of the foregoing, where the number of quantics has been limited to two.

So far, forms in a single set of variables have been treated. In addition to these, the theory of algebraic forms must deal with those containing two or more independent, or cogredient, or contragredient sets of variables; first of all, if we follow Clebsch, those containing two sets of the same number of variables, mutually contragredient. Such mixed forms he calls *connexes*, and speaks of them by the use of two indices, as (m_1, m_2), denoting respectively the order in the one set and the order in the other set of variables. Without going into details, I may offer here the outline statement that,

If two connexes have the indices (m_1, m_2) *of the one not less than, and at least one of them greater than the corresponding indices* (n_1, n_2), *respectively, of the other; then there exists in the aggregate of their simultaneous covariants a subgroup, their semi-combinants; and their semi-combinant groundforms and affiliants can be found by a method exactly analogous to that here given for quantics in a single set of variables.*

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